M.Phil/Pre Ph.D Regular & Supplementary Examinations – OCTOBER, 2023
R211003B
Paper –III: Advanced Algebra.
MATHEMETICS

Time : 3 hrs Maximum Marks : 100

Answer One Question from Each Unit
All Questions Carry Equal Marks

UNIT-I

1) (a) Let $F \subseteq E \subseteq K$ be fields. If $[K : E] < \infty$ and $[E : F] < \infty$, then prove that
(i) $[K : F] < \infty$
(ii) $[K : F] = [K : E][E : F]$

(b) Let E be an algebraic extension of a field F and let $\sigma: F \to L$ be an embedding of F Into an algebraically closed field L. Then prove that σ can be extended to an embedding $\eta: E \to L$.

(or)

2) (a) Prove that the prime field of a field F is either isomorphic to Q or to Z/(p), p prime.
(b) Prove that the multiplicative group of nonzero elements of a finite field is cyclic.

UNIT-II

3) (a) Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of A-modules. Then prove that M is Noetherian if and only if M' and M'' are Noetherian.
(b) Suppose that M has a composition series of length n. Then prove that every composition series of M has length n, and every chain in M can be extended to a composition series.

(or)

4) Prove that for k-vector spaces V, the following conditions are equivalent:
(i) Finite dimension
(ii) Finite length
(iii) a.c.c
(iv) d.c.c

Moreover, if the conditions are satisfied, length equal to dimension.

UNIT-III

5) (a) Prove that in an Artin ring the nilradical N is nilpotent.
(b) Let A be a Noetherian local ring, M its maximal ideal. Then prove that exactly one of the following two statements is true:
(i) $M^n \neq M^{n+1}$ for all n
(ii) $M^n = 0$ for some n, in which case A is an Artin local ring.

(or)

6) Let A be an Artin local ring. Then prove that the following are equivalent:
   (i) Every ideal A is principle
   (ii) The maximal ideal m is principle
   (iii) $\text{Dim}_k(m/m^2) \leq 1$.

UNIT-IV

7) State and prove that the Hilbert's Basis Theorem.

(or)

8) (a) Prove that in a Noetherian ring every irreducible ideal is primary.
   (b) Let $I \neq \langle 1 \rangle$ be an ideal in a Noetherian ring A. Then prove that the prime ideals which belong to I are precisely the prime ideals which occur in the set of ideals $(I : x)(x \in A)$

UNIT-V

9) (a) Prove that if r(I) is maximal then I is primary. In particular, the powers of a maximal ideal M are M-primary.
   (b) Let I be a decomposable ideal. Then prove that any prime ideal $P \supseteq I$ contains a minimal prime ideal belonging to I and thus the minimal prime ideals of I are precisely the minimal elements in the set of all prime ideals containing I.

(or)

10) (a) Let S be a multiplicatively closed subset of A and let Q be a P-primary ideal. Then prove the following:
   (i) If $S \cap P \neq \emptyset$, then $S^{-1}Q = S^{-1}A$
   (ii) If $S \cap P = \emptyset$, then $S^{-1}Q$ is $S^{-1}P$-primary and its contraction in A is Q.

   (b) State and prove that the 2^{nd} uniqueness theorem.

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